4.2 Metric Spaces.

4.2A The properties of the absolute-value function

$|0| = 0$ — ①

$|a| > 0 \quad (a \in R,\ a \neq 0)$, — ②

$|a| = |-a| \quad (a \in R)$ — ③

$|a+b| \leq |a| + |b| \quad (a, b \in R)$ — ④

For $x, y \in R$, the geometric interpretation of $|x-y|$ is the distance from x to y. If we define the "distance function" ρ by
$$\rho(x,y) = |x-y| \quad (x, y \in R)$$
then the properties ① to ④ have the following consequences for any points $x, y, z \in R$

$$\rho(x,x) = 0 \quad \text{— ⑤}$$

(That is, the distance from a point to itself is 0)

$$\rho(x,y) > 0 \quad (x \neq y) \quad \text{— ⑥}$$

((6) The distance between two distinct points is strictly positive)

$$\rho(x,y) = \rho(y,x)$$

(The distance from x to y is equal to the distance from y to x)

$$\rho(x,y) \leq \rho(x,z) + \rho(z,y) \quad \text{(triangle inequality)}$$

4.2B Definition:
Let M be any set. A metric for M is a function ρ with domain $M \times M$ and range contained in $[0, \infty)$ such that

$\rho(x,x) = 0 \quad (x \in M)$ — ⑤

$\rho(x,y) > 0 \quad (x, y \in M,\ x \neq y)$, — ⑥

$\rho(x,y) = \rho(y,x) \quad (x, y \in M)$ — ⑦

$\rho(x,y) \leq \rho(x,z) + \rho(z,y) \quad (x, y, z \in M)$

Triangle inequality — ⑧

If ρ is a metric for M, then the ordered pair ② $\langle M, \rho \rangle$ is called a metric space.

A metric for M thus has all the properties ⑤ to ⑧ of the distance function $|x-y|$ for R.

Example : 1 The function ρ defined by $\rho(x, y) = |x-y|$ is obviously a metric for the set R of real numbers. we denote the resulting metric space $\langle R, \rho \rangle$ by R^1. we call this metric ρ the absolute value metric.

Ex 2: Here is another metric for the set R. Define

$d : R \times R \to [0, \infty)$ by

$$d(x, x) = 0 \quad (x \in R)$$

$$d(x, y) = 1 \quad (x, y \in R, \ x \neq y).$$

That is the distance $d(x, y)$ between any two distinct points $x, y \in R$ is 1. The metric d is called the discrete metric.

4·3 Limits in metric spaces.

4·3 A Defn: We say that $f(x)$ approaches L (where $L \in M_2$) as x approaches a if given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\rho_2(f(x), L) < \varepsilon \qquad (0 < \rho_1(x, a) < \delta)$$

In this case we write $\lim_{x \to a} f(x) = L$,

(or)

$f(x) \to L$ as $x \to a$.

4·3B Theorem:

Let $\langle M, \rho \rangle$ be a metric space and let 'a' be a point in M. Let f and g be real-valued functions, whose domains are subsets of M. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = N$, then

(i) $\lim_{x \to a} [f(x) + g(x)] = L + N$.

(ii) $\lim_{x \to a} [f(x) - g(x)] = L - N$.

(iii) $\lim_{x \to a} (f(x) g(x)) = LN$,

and if $N \neq 0$ (iv) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{N}$.

proof(i) Given $\lim_{x \to a} f(x) = L$

$\therefore$ By defn given $\epsilon > 0$ there exists $\delta_1 > 0$ such that $|f(x) - L| \cancel{\rho(f(x), L)} < \frac{\epsilon}{2}$ $\quad 0 < \rho(x, a) < \delta_1$ — (1)

Also given $\lim_{x \to a} g(x) = N$

By defn given $\epsilon > 0$ there exists $\delta_2 > 0$ such that $|g(x) - N| < \frac{\epsilon}{2}$ $\quad 0 < \rho(x, a) < \delta_2$ — (2)

let $\delta = \min\{\delta_1, \delta_2\}$

when $0 < \rho(x, a) < \delta$ then

$|f(x) - L| < \epsilon/2$ and $|g(x) - N| < \epsilon/2$ — (3)

now when $0 < \rho(x, a) < \delta$

consider

$$|(f(x) + g(x)) - (L + N)| = |(f(x) - L) + (g(x) - N)|$$

$$\leq |f(x) - L| + |g(x) - N|$$

$$< \epsilon/2 + \epsilon/2$$

$\Rightarrow |(f(x) + g(x)) - (L + N)| < \epsilon$ when $0 < \rho(x, a) < \delta$

$\Rightarrow \lim_{x \to a} (f(x) + g(x)) = L + N$.

Proof (ii) when $0 < \rho(x,a) < \delta$

$\therefore$ By (3) $|f(x)-L| < \varepsilon_1/2$ and $|g(x)-N| < \varepsilon_1/2$ — (3)

Consider

$$\left|(f(x)-g(x))-(L-N)\right| = \left|(f(x)-L)+(N-g(x))\right|$$
$$\leq |f(x)-L| + |N-g(x)|$$
$$\leq |f(x)-L| + |g(x)-N|$$
$$< \varepsilon_1/2 + \varepsilon_1/2$$

$|(f(x)-g(x))-(L-N)| < \varepsilon_1$ when $0 < \rho(x,a) < \delta$

$\Rightarrow \lim\limits_{x\to a}(f(x)-g(x)) = L-N$ proof (ii)

Proof (iii) To prove $\lim\limits_{x\to a} f(x)g(x) = LN$.

Given $\lim\limits_{x\to a} g(x) = N$, given $\varepsilon_1 > 0$ there exists $\delta_1 > 0$

Such that $|g(x)-N| < 1$ $0 < \rho(x,a) < \delta_1$ $\because \varepsilon_1 < 1$

Thus $|g(x)| < |N|+1 = Q$ $0 < \rho(x,a) < \delta_1$ — (4)

Now $f(x)g(x) - LN = f(x)g(x) - Lg(x) + Lg(x) - LN$

$f(x)g(x) - LN = g(x)(f(x)-L) + L(g(x)-N)$

$$|f(x)g(x) - LN| \leq |g(x)|\,|f(x)-L| + |L|\,|g(x)-N|$$

Hence if $0 < \rho(x,a) < \delta$

$|f(x)g(x) - LN| \leq Q \cdot |f(x)-L| + |L|\,|g(x)-N|$ — (5)

by (4)

$\lim\limits_{x\to a} f(x) = L$ $\therefore$ By defn given $\varepsilon_1 > 0$ $\exists$ $\delta_2 > 0$ such that $|f(x)-L| < \dfrac{\varepsilon_1}{2Q}$, $0 < \rho(x,a) < \delta_2$ — (6)

and there exists $\delta_3 > 0$ such that

$$|g(x) - N| < \frac{\varepsilon}{2|L|} \quad 0 < \rho(x, a) < \delta_3 \quad \text{— } 7$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ then from 5, 6, 7

$$|f(x)g(x) - LN| < \varepsilon \quad 0 < \rho(x, a) < \delta$$

$$\Rightarrow \lim_{x \to a} f(x)g(x) = LN.$$